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Transition Amplitude Spaces and Quantum Logics with Vector-Valued States

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Relations between transition amplitude spaces and quantum logics are studied. It is shown that transition amplitude spaces correspond to quantum logics with rich enough sets of vector-valued states.

1. INTRODUCTION

The notion of a transition amplitude space (tas) was introduced in Gudder and Pulmannová (1987) (see also Gudder, 1988). It is defined as follows: Let S be a nonempty set and let $A: S\times S\to C$. We say that $x, y\in S$ are orthogonal ($a\perp b$) if $x\neq y$ and $A(x, y)=0$. We call a set $M\subset S$ an A-set if for every $x, y\in S$,

$$\sum_{z\in M} |A(x, z)\bar{A}(y, z)|<\infty,$$

and

$$A(x, y)=\sum_{z\in M} A(x, z)\bar{A}(y, z)$$

Denote the collection of A-sets by N_A. We call $A: S\times S\to C$ a transition amplitude if (i) $N_A\neq\varnothing$ and (ii) $A(x, x)=1$. If A is a transition amplitude, we call (S, A) a transition amplitude space. We then have $A(x, y)=\bar{A}(y, x)$. A strong (ultrastrong) tas is a tas (S, A) which satisfies $A(x, y)=1\Rightarrow x=y$ $[|A(x, y)|=1\Rightarrow x=y]$. A tas (S, A) is total if every maximal orthogonal subset of S is an A-set. Gudder and Pulmannová (1987) proved that every tas admits a representation, i.e., there exist a Hilbert space H and a map $\phi: S\to H$ such that $A(x, y)=(\phi(x), \phi(y))$ for every $x, y\in S$. The map ϕ is injective if and only if the tas is strong.

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Let (S, A) be a total tas. Define a relation R on S by xRy if $|A(x, y)| = 1$. Then R is an equivalence relation and $(S/R, T)$, where $T(x, y) = |A(x, y)|^2$, is a transition probability space (Gudder and Pulmannová, 1987, Theorem 2.2).

Recall that a couple (S, T) is a transition probability space (tps) if S is a nonempty set and $T: S \times S \to \langle 0, 1 \rangle$ satisfies:

(i) $T(x, y) = 1 \Leftrightarrow x = y$.

(ii) $T(x, y) = T(y, x)$ for any $x, y \in S$

(iii) Calling x and y orthogonal $(x \perp y)$, if $T(x, y) = 0$, we have $\sum_{y \in M} T(x, y) = 1$ for every maximal set M of pairwise orthogonal elements of S and every $x \in S$. A map T with the properties (i)–(iii) is called a transition probability and (S, T) is a transition probability space (Mielnik, 1968, 1969).

Recall that a quantum logic is a partially ordered set L with 0 and 1, with an orthocomplementation $': L \to L$ such that:

(i) $(a')' = a$.

(ii) $a \leq b \Rightarrow b' \leq a'$.

(iii) $a \vee a' = 1$, $a \wedge a' = 0$.

(iv) Calling $a, b \in L$ orthogonal if $a \leq b'$, $\bigvee_{i \in N} a_i$ exists in L for every sequence $(a_i)_{i \in N}$ of pairwise orthogonal elements of L.

(v) $a \leq b \Rightarrow b = a \vee (a' \wedge b)$.

A logic L is orthocomplete if $\vee_{i \in I} a_i$ exists in L for any set $(a_i)_{i \in I}$ of pairwise orthogonal elements in L. A state on L is a map $s: L \to \langle 0, 1 \rangle$ such that $s(1) = 1$ and $s(a \vee b) = s(a) + s(b)$ for any $a, b \in L$, $a \perp b$. A state s on L is σ-additive (completely additive) if $s(\vee_{i \in N} a_i) = \sum_{i \in N} s(a_i)$ for any sequence $(a_i)_{i \in N}$ of pairwise orthogonal elements of L [$s(\vee_{i \in I} a_i) = \sum_{i \in I} s(a_i)$ for any set $(a_i)_{i \in I}$ of pairwise orthogonal elements of L such that $\vee_{i \in I} a_i$ exists in L].

An element $a \in L$ is a carrier of a state s if $s(b) = 0 \Leftrightarrow b \perp a$. An element $x \in L$ is an atom if $y \in L$, $y \leq x \Rightarrow y = 0$ or $y = x$. A logic L is atomistic if every element in L is the supremum of all atoms lying under it [see Gudder (1979), Beltrametti and Cassinelli (1981), and Varadarajan (1985) for the basic facts about quantum logics].

Relations between transition probability spaces and quantum logics have been studied by several authors (e.g., Belinfante, 1976; Bugajska, 1974; Deliyannis, 1984; Pulmannová, 1986*a*,*b*). It has been shown that transition probability spaces [with a weaker symmetry property $T(x, y) = 0 \Leftrightarrow T(y, x) = 0$ replacing the symmetry condition $T(x, y) = T(y, x)$] are in one-to-one correspondence (up to isomorphisms) with atomistic, orthocomplete quantum logics such that to every atom a of the logic there is a unique completely additive state with the carrier a. If (S, T) is a tps, the corresponding quantum logic is constructed as follows. For any $A \subset S$, put $A^\perp = \{y \in S \mid T(x, y) = 0$ for all $x \in A\}$, and $\bar{A} = A^{\perp\perp}$. Then the logic $L = \{\bar{E} \mid E$ is an orthogonal subset of $S\}$. For any $x \in S$, $\bar{x} = x$, and x is an atom of L. In addition, for every $x \in S$, the map $s_x : L \to \langle 0, 1\rangle$ defined by $s_x(\bar{E}) = \sum_{y \in E} T(x, y)$ defines the state with the carrier x (Deliyannis, 1984; Pulmannová, 1986*a*,*b*).

2. A QUANTUM LOGIC CHARACTERIZATION OF TAS'S

We shall investigate a class of quantum logics which characterize total transition amplitude spaces. Let L be a quantum logic and let H be a Hilbert space. An H-valued (σ-additive) state on L is a map $\xi : L \to H$ such that:

(i) For any $a, b \in L$, $a \perp b$ we have $(\xi(a), \xi(b)) = 0$ and $\xi(a \vee b) = \xi(a) + \xi(b)$.

(ii) For any sequence $(a_i)_{i \in N}$ of pairwise orthogonal elements of L we have $\xi(\vee_{i \in N} a_i) = \sum_{i \in N} \xi(a_i)$, where the sequence converges in norm in H.

(iii) $|\xi(a)|^2 = 1$.

An H-valued state ξ on L is completely additive if for any set $(a_i)_i$ of pairwise orthogonal elements of L such that $\vee_i a_i$ exists in L, we have $\xi(\vee_i a_i) = \sum_i \xi(a_i)$.

It is clear that if ξ is an H-valued state on L, then the map $a \mapsto \|\xi(a)\|^2$ is a real-valued state on L, which is completely additive if and only if ξ is completely additive.

H-valued states on logics have been studied by several authors (e.g., Dvurečenskij and Pulmannová, 1981; Hamhalter and Pták, 1989; Jajte and Pazskiewcz, 1978; Kruszynski, 1988; Mayet, 1987). The following statement, proved in Kruszynski (1988), will be especially useful in the sequel. Before formulating it, we need several definitions and remarks.

Let ξ, η be H-valued states on L. We say that ξ and η are biorthogonal if for any a, $b \in L$, $a \perp b$, we have $\xi(a) \perp \eta(b)$. It is easy to check that ξ and η are biorthogonal if and only if $(\alpha\xi + \beta\eta)(\|\alpha\xi(1) + \beta\eta(1)\|^{-2})$ is also an H-valued state on L for any $\alpha, \beta \in C$.

A family $\mathcal{N}$ of H-valued states is a biorthogonal family if every two states $\xi, \nu \in \mathcal{N}$ are biorthogonal. A biorthogonal family $\mathcal{N}$ is maximal if every H-valued state on L which is biorthogonal with every element of $\mathcal{N}$ belongs to $\mathcal{N}$. Owing to the Zorn lemma, any biorthogonal family is contained in a maximal one.

Theorem 1 (Kruszynski, 1988). Let $\mathcal{N}$ be a biorthogonal family of H-valued states on a logic L. Then there is a map $\Psi: L \to L(H)$ [where $L(H)$ is the projection logic of H] such that:

(i) $\Psi(a)\xi(a) = \xi(a)$ for every $\xi \in \mathcal{N}$ and all $a \in L$.

(ii) $a, b \in L$, $a \leq b \Rightarrow \Psi(a) \leq \Psi(b)$.

(iii) $a, b \in L$, $a \perp b \Rightarrow \Psi(a) \perp \Psi(b)$ and for every sequence $(a_i)_{i \in N}$ of orthogonal elements of L, $\Psi(\vee_{i \in N} a_i) = \sum_{i \in N} \Psi(a_i)$, where the series converges in strong operator topology on H. If, in addition, all the elements in $\mathcal{N}$ are completely additive, $\Psi(\vee_i a_i) = \sum_i \Psi(a_i)$ for any orthogonal set $(a_i)_i \subset L$.

(iv) To every $\xi \in \mathcal{N}$, there exists a vector $v_\xi \in H$ such that $\xi(a) = \Psi(a)v_\xi$, $a \in L$.

Note that $\Psi(a)$ is the projection onto the closed subspace $\mathcal{N}_0(a) = \{c \cdot \xi(a) \mid \xi \in \mathcal{N}_0, c \in C\}$, where $\mathcal{N}_0$ is a maximal biorthogonal family containing $\mathcal{N}$. As a consequence of the above theorem, we obtain that the map Ψ is a orthohomomorphism of L into $L(H_0)$, where H_0 is the subspace corresponding to $\Psi(1)$. As $\Psi(a)v_\xi = \Psi(a)\Psi(1)v_\xi$, we may suppose that $v_\xi \in H_0 (\xi \in \mathcal{N})$.

Theorem 2. Let (S, T) be a transition probability space. Then there is a transition amplitude space (S, A) such that $T(x, y) = |A(x, y)|^2$, $x, y \in S$, if and only if there is a Hilbert space H and a map $\Phi: S \to H$ such that $T(x, y) = |(\Phi(x), \Phi(y))|^2$ for any $x, y \in S$.

Proof. Let (S, A) be a tas such that $T(x, y) = |A(x, y)|^2$, $x, y \in S$. As every tas admits a representation, there is a Hilbert space H and a map $\Phi: S \to H$ such that $T(x, y) = |A(x, y)|^2 = |(\Phi(x), \Phi(y))|^2$ for any $x, y \in S$.

Now let there be a Hilbert space H and a map $\Phi: S \to H$ such that $T(x, y) = |(\Phi(x), \Phi(y))|^2$, $x, y \in S$. Put $A(x, y) = (\Phi(x), \Phi(y))$, $x, y \in S$. Then $A(x, x) = \|\Phi(x)\|^2 = [T(x, x)]^{1/2} = 1$. Let $M \subset S$ be any maximal orthogonal subset. Then $\Phi(M)$ is an orthogonal subset of H^1, and since

$$1 = \sum_{y \in M} T(x, y) = \sum_{y \in M} |(\Phi(x), \Phi(y))|^2 = \|\Phi(x)\|^2$$

for every $x \in S$, we get $\|\Phi(x) - \sum_{y \in M} (\Phi(x), \Phi(y))\Phi(y)\| = 0$. Hence $\Phi(x) = \sum_{y \in M} (\Phi(x), \Phi(y))\Phi(y)$ for every $x \in S$. Therefore,

$$\begin{aligned} A(x, y) &= (\Phi(x), \Phi(y)) \\ &= \sum_{z \in M} (\Phi(x), \Phi(z))(\Phi(z), \Phi(y)) \\ &= \sum_{z \in M} A(x, z)A(z, y) \\ &= \sum_{z \in M} A(x, z)\bar{A}(y, z) \end{aligned}$$

Hence M is an A-set. This proves that (S, A) is a tas. ∎

Now we are able to state and prove our main result.

Theorem 3. Let (S, T) be a (symmetric) transition probability space and let L be the corresponding logic. Then there is a total transition amplitude space (S, A) such that $T(x, y) = |A(x, y)|^2$ if and only if there is a Hilbert space H such that:

(i) To every $x \in S$, there is an H-valued state $\xi_x : L \to H$ such that $s_x(a) = \|\xi(a)\|^2$, $a \in L$, where s_x is the unique state on L with the carrier x.

(ii) The set $\{\xi_x | x \in S\}$ forms a biorthogonal family of H-valued states on L.

(iii) The set $\{\xi_z(1) | z \in M\}$ is an orthogonal base of H for any maximal orthogonal subset M of S.

Proof. (1) Let (S, A) be a total tas such that $T(x, y) = |A(x, y)|^2$, $x, y \in S$. Let L be the logic corresponding to (S, T). It is easy to check that L is isomorphic to the event structure of the tas (S, A) (Gudder and Pulmannová, 1987, Theorem 4.10; Pulmannová and Gudder, 1987, Corollary 1.4). Let $\Phi : S \to H$ be a representation of (S, A). Then $A(x, y) = (\Phi(x), \Phi(y))$, $x, y \in S$. This representation yields an orthohomomorphism $\Psi : L \to L(H)$ [see Gudder and Pulmannová (1987), remarks before Theorem 4.10]. For every $x \in S$, define $\xi_x(a) = \Psi(a)\Phi(x)$, $a \in L$. Then $\xi_x : L \to H$ is an H-valued state on L. Let $a \in L$ and let $a = \bar{E}$ (i.e., E is a maximal set of orthogonal atoms contained in a—the event corresponding to a). Then we have

$$\begin{aligned} s_x(a) &= \sum_{y \in E} T(x, y) \\ &= \sum_{y \in E} |(\Phi(x), \Phi(y))|^2 \\ &= \sum_{y \in E} (P_{\Phi(y)} \Phi(x), \Phi(x)) \\ &= \|(\Psi(a)\Phi(x))\|^2 \\ &= \|\xi_x(a)\|^2 \end{aligned}$$

[Here $P_{\Phi(y)}$ denotes the projection on the one-dimensional subspace of H generated by $\Phi(y)$. We have $\sum_{y \in E} P_{\Phi(y)} = \Psi(a)$.] This proves (i) of the theorem. Part (ii) is straightforward. To prove (iii), let M be a maximal orthogonal subset of S. For any $z \in M$, we have $\xi_z(1) = \Psi(1)\Phi(z) = \Phi(z)$. As $\Phi : \mathrm{S} \to \mathrm{H}$ is a representation, $\Phi(M)$ is a base in H.

(2) Let (S, T) be a tps, L be the corresponding logic, and suppose that (i)–(iii) are satisfied. By Theorem 1, there is an orthohomomorphism $\Psi : L \to L(H)$, such that $\xi_x(a) = \Psi(a)v_x$, where v_x is a vector in H. We show that

$\Psi(x)$ is one-dimensional for every $x \in S$. (We identify the atoms in L with the elements of S.) Indeed, let M be a maximal orthogonal subset of S such that $x \in M$. As $1 = s_x(x) = \|\xi_x(x)\|^2 = \|\Psi(x)v_x\|^2$, we have $v_x \in \Psi(x)$. Let $v \in \Psi(x)$, $v \perp v_x$. For any $z \in M$, $z \neq x$, we have $\|\Psi(x)v_z\|^2 = s_z(x) = T(z, x) = 0$; hence $\Psi(x) \perp v_z$, and hence $v_z \perp v$. Now $\xi_z(1) = \Psi(1)v_z = v_z$; hence $\{v_z \mid z \in M\}$ is a base in H. This implies that $v = 0$. Hence $\Psi(x)v_y = (v_y, v_x)v_x$. For $x, y \in S$, define $A(x, y) = (\xi_x(1), \xi_y(1)) = (v_x, v_y)$. We obtain

$$|A(x, y)|^2 = |(v_x, v_y)|^2 = \|\Psi(y)v_x\|^2 = \|\xi_x(y)\|^2 = s_x(y) = T(x, y)$$

If we put $\Phi(x) = v_x$, we obtain by Theorem 2 that (S, A) is a tas. ■

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